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UPPER BOUNDS FOR THE EFFECTIVE THERMAL CONTACT RESISTANCE BETWEEN BODIES WITH ROUGH SURFACES

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Abstract—In a previous paper [*Int. J. Heat Mass Transfer* **25**, 111–117 (1982)] variational principles had been given from which upper and lower bounds for the effective thermal contact resistance may be deduced. In the present paper this method is used to determine upper bounds for arbitrary geometry and distribution of the contact spots. The bounds are functionals of the two-point correlation function of the local contact resistance. Special attention has been paid to a binary model admitting only two discrete values for the local contact resistance inside and outside the direct contact areas, respectively. The constriction resistance is mainly characterized by a correlation length of the contact spots. The results are compared to a special bound for circular contact spots obtained in the earlier work.

NOMENCLATURE

A,	area;
a,	area fraction;
B,	operator defined by equation (3.6);
b,	stochastic function of position defined by equation (3.7);
b_i,	constant defined by equation (5.3);
f,	arbitrary function of position;
G_0,	inversion operator of L_0;
g,	pair correlation function of the circle centres;
J,	integral operator defined by equation (1.3);
J_1,	Bessel function of first order;
K,	thermal conductivity;
$\mathbf{k}$,	vector of Fourier space;
L_0,	non-stochastic operator defined by equation (3.1);
l_c,	correlation length defined by equation (2.13);
$\tilde{l}_c$,	characteristic length defined by equation (5.8);
N,	number of single contacts;
n,	area density of contacts;
P,	projection operator defined by equation (1.4);
p,	pair distribution function of the circle centres;
Q,	heat flow;
q,	z-component of the heat flow density;
$\tilde{q}$,	trial function of q;
q_0,	constant heat flow density;
$\mathbf{R}$,	2-dim. position vector (x, y);
R_{eff},	effective thermal contact resistance;
R_0,	radius of a circular contact spot;
T_0,	constant temperature jump;
ΔT,	jump in temperature;
$\widetilde{\Delta T}$,	trial function of ΔT;
w_{st},	random local contact resistance;
w_{eff},	effective value of w_{st};
w_+, w_-,	upper and lower bound of w_{eff}, respectively;
w_i, w_e,	local contact resistance inside and outside a contact, respectively;
w_0,	constant parameter;
w',	stochastic part of w_{st} defined by equation (3.1).

Greek symbols

Γ_w, Γ,	correlation functions defined by equations (2.5) and (2.14), respectively;
γ,	variational parameter;
γ_0,	optimal value of γ;
Θ, Θ^c,	step functions defined by equations (2.8) and (6.2), respectively;
φ, φ_1,	given functions defined by equations (6.11) and (6.13), respectively.

1. INTRODUCTION

THIS paper is devoted to the estimation of the thermal or electric contact resistance between two bodies with rough surfaces. The phenomenon of contact resistance is mainly caused by two effects. The first one is due to the surface roughness, i.e. the two bodies are directly connected only at some spots, and, therefore, the heat flow is constricted near these spots. This contribution to the macroscopic contact resistance is called constriction resistance. The second effect—the local contact resistance at the contact spots—is strongly influenced by the presence of oxide or liquid layers. Therefore, we shall call it film resistance. This quantity is assumed to be known and enters as a parameter into the present calculations. Our aim is to determine the constriction resistance.

Previous estimations of the constriction resistance have been based on special models for the geometry of the rough surfaces. In most of them a regular arrangement of single circular contact spots is considered. In practice, however, there exist very different arrangements and various shapes of the contact spots and it is not clear to what extent the results obtained for special models are valid in more general cases.

In a previous paper [1] the authors proposed an alternative approach to this problem. Instead of calculating approximate values, one may determine rigorous lower and upper bounds for the effective contact resistance by means of variational principles. The method has been developed for the following model. The rough surfaces are approximated by the plane $z = 0$. The heat transport across this plane is determined by a random local contact resistance defined by

$$w_{st}(\mathbf{R}) = (K/2)\frac{\Delta(\mathbf{R})}{q(\mathbf{R})}, \quad \mathbf{R} = (x, y) \tag{1.1}$$

where K denotes the thermal conductivity of both bodies in contact. The local contact resistance w_{st} describes the geometry and the positions of the contact areas as well as the film resistance caused, for instance, by surface layers. In general, $w_{st}(\mathbf{R})$ takes low values inside the areas of direct contact and high values outside them. Due to the strong variations of w_{st}, the jump in temperature at the surfaces $\Delta T(\mathbf{R})$ as well as the heat flow density $q(\mathbf{R})$ are strongly fluctuating functions of position. Besides equation (1.1) these two quantities are connected by the following integral equation derived in [1]

$$P\Delta T(\mathbf{R}) = (2/K)\,JPq(\mathbf{R}) \tag{1.2}$$

where the integral operator J, applied to an arbitrary function f, is defined by

$$Jf(\mathbf{R}) = \int \mathrm{d}^2\mathbf{R}' \frac{1}{2\pi R'} f(\mathbf{R} + \mathbf{R}'). \tag{1.3}$$

The projection operator P yields the deviations of a function f from its mean value $\langle f \rangle$

$$Pf(\mathbf{R}) = f(\mathbf{R}) - \langle f(\mathbf{R})\rangle \tag{1.4}$$

where the angular brackets denote an average over an area A very large compared to the single contact spots

$$\langle f(\mathbf{R})\rangle = \frac{1}{A}\int_A \mathrm{d}^2\mathbf{R} f(\mathbf{R}). \tag{1.5}$$

In the following we always take the limit of an infinite plane $A \to \infty$. The effective thermal contact resistance R_{eff} measured in a macroscopic experiment is given by

$$R_{eff} = \frac{\langle \Delta T\rangle}{Q} \tag{1.6}$$

with the total heat flow

$$Q = \int_A \mathrm{d}^2\mathbf{R}\, q(\mathbf{R}) = A\langle q\rangle. \tag{1.7}$$

Analogously to equation (1.1) we can define an effective quantity w_{eff}

$$w_{eff} = \frac{K}{2}\frac{\langle \Delta T\rangle}{\langle q\rangle} \tag{1.8}$$

$$R_{eff} = \frac{2}{KA} w_{eff}. \tag{1.9}$$

By the aid of variational principles given in ref. [1], we may derive various upper and lower bounds w_+ and w_- confining the actual value of w_{eff} (or R_{eff}) to the interval

$$w_+ \geq w_{eff} \geq w_-. \tag{1.10}$$

The bounds involve statistical information about the local contact resistance w_{st}. To get a good estimation of w_{eff}, the bounds should be as narrow as possible.

In this paper we only deal with general upper bounds which can be obtained from the following variational principle [1]

$$w_+ = \langle \tilde{q}\rangle^{-2}\langle \tilde{q}(w_{st} + JP)\tilde{q}\rangle \geq w_{eff}. \tag{1.11}$$

For every $\tilde{q}$ equation (1.11) yields an upper bound on w_{eff}. Equality holds for the exact heat flow density $\tilde{q} = q(\mathbf{R})$ satisfying the stochastic integral equation

$$(w_{st} + JP)q = \langle w_{st}\, q\rangle = w_{eff}\langle q\rangle. \tag{1.12}$$

In order to derive a useful bound we have to choose a trial function $\tilde{q}(\mathbf{R})$ which fits the exact field $q(\mathbf{R})$ sufficiently well.

In section 2 we start with very simple trial functions for the heat flow density to deduce bounds according to equation (1.11). To get a better bound, a more sophisticated trial function is constructed in section 3. By means of this function a general expression for an upper bound including only the two-point moment of the stochastic field $w_{st}(\mathbf{R})$ is derived in section 4. In section 5 this bound is specified to a binary model for the local contact resistance. Finally in section 6 a comparison is made with a bound previously obtained for an arbitrary distribution of circular contact spots [1]. Numerical results are drawn for a triangular lattice.

2. SIMPLE UPPER BOUNDS

To calculate an upper bound according to equation (1.11) we have to choose a trial function $\tilde{q}(\mathbf{R})$ for the heat flow density. The simplest approximation is a constant field

$$\tilde{q}(\mathbf{R}) = q_0 \tag{2.1}$$

which yields

$$w_+ = \langle w_{st}(\mathbf{R})\rangle. \tag{2.2}$$

In general this bound is not very satisfactory. In the case of an infinite local contact resistance outside the contact spots this bound even diverges.

We find a more reasonable bound if we choose the trial function $\tilde{q}$ in such a way that the corresponding temperature jump ΔT becomes constant, i.e.

$$\widetilde{\Delta T}(\mathbf{R}) = T_0;\ \tilde{q}(\mathbf{R}) = \frac{K}{2}\frac{\widetilde{\Delta T}(\mathbf{R})}{w_{st}(\mathbf{R})} = \frac{KT_0}{2w_{st}(\mathbf{R})}. \tag{2.3}$$

Inserting this expression into equation (1.11) we get the following bound

$$w_+ = \left\langle \frac{1}{w_{st}} \right\rangle^{-1} + \left\langle \frac{1}{w_{st}} \right\rangle^{-2} \int d^2\mathbf{R}' \times \frac{1}{2\pi R'} \left\langle \frac{1}{w_{st}(\mathbf{R})} P \frac{1}{w_{st}(\mathbf{R}+\mathbf{R}')} \right\rangle$$

$$= \left\langle \frac{1}{w_{st}} \right\rangle^{-1} + \int d^2\mathbf{R}' \frac{1}{2\pi R'} \Gamma_w(\mathbf{R}') \tag{2.4}$$

where

$$\Gamma_w(\mathbf{R}') = \left\langle \frac{1}{w_{st}} \right\rangle^{-2} \left\langle \frac{1}{w_{st}(\mathbf{R})} P \frac{1}{w_{st}(\mathbf{R}+\mathbf{R}')} \right\rangle$$

$$= \left\langle \frac{1}{w_{st}} \right\rangle^{-2} \left\langle \frac{1}{w_{st}(\mathbf{R})} \frac{1}{w_{st}(\mathbf{R}+\mathbf{R}')} \right\rangle - 1. \tag{2.5}$$

According to equation (1.5) the angular brackets always mean an average with respect to the position $\mathbf{R}$. The bound (2.4) contains only partial information about the local contact resistance w_{st}. Besides the mean value of $1/w_{st}$, only the two-point moment of this stochastic quantity is involved. It characterizes the correlation between the values of $1/w_{st}$ at two different points separated by the vector $\mathbf{R}'$.

In order to illustrate the meaning of the bound (2.4) let us consider the special example of a 'binary' model in which the stochastic parameter w_{st} takes only two values w_i and w_e inside and outside the contact spots, respectively

$$w_{st}(\mathbf{R}) = w_i\Theta(\mathbf{R}) + w_e(1 - \Theta(\mathbf{R})) = (w_i - w_e)\,\Theta(\mathbf{R}) + w_e, \tag{2.6}$$

$$\frac{1}{w_{st}(\mathbf{R})} = \left(\frac{1}{w_i} - \frac{1}{w_e}\right)\Theta(\mathbf{R}) + \frac{1}{w_e}, \tag{2.7}$$

where the step function Θ is defined by

$$\Theta(\mathbf{R}) = \begin{cases} 1 & \text{for } \mathbf{R} \text{ in a contact spot,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

This model represents a good approximation to the actual situation in many practical cases. Inserting equation (2.7) into the bound (2.4), we obtain

$$\Gamma_w(\mathbf{R}') = \left(\frac{w_e - w_i}{\tilde{w}}\right)^2 (\langle\Theta(\mathbf{R})\Theta(\mathbf{R}+\mathbf{R}')\rangle - a^2) = \left(\frac{w_e - w_i}{\tilde{w}}\right)^2 a(1-a)\,\Gamma(\mathbf{R}') \tag{2.9}$$

$$w_+ = \frac{w_e w_i}{\tilde{w}} + \left(\frac{w_e - w_i}{\tilde{w}}\right)^2 a(1-a)\,l_c \tag{2.10}$$

where

$$\tilde{w} := a w_e + (1-a)\,w_i \tag{2.11}$$

$$a := \langle\Theta(\mathbf{R})\rangle \tag{2.12}$$

$$l_c := \int d^2\mathbf{R}' \frac{1}{2\pi R'} \Gamma(\mathbf{R}') \tag{2.13}$$

$$\Gamma(\mathbf{R}') := \frac{1}{a(1-a)}[\langle\Theta(\mathbf{R})\Theta(\mathbf{R}+\mathbf{R}')\rangle - a^2]. \tag{2.14}$$

The symbol a denotes the area fraction of the contact spots. The function $\Gamma(\mathbf{R}')$ depends only on the geometry of the contact area. It is defined in such a manner that

$$\Gamma(0) = 1, \qquad \lim_{|\mathbf{R}'|\to\infty} \Gamma(\mathbf{R}') = 0. \tag{2.15}$$

The latter equation expresses the fact that for large distances there is no correlation between the contact areas. A rough measure for the range of the correlation function $\Gamma(\mathbf{R}')$ is given by the parameter l_c which, therefore, we shall call correlation length. In general, it is of the same order as the size of a single contact spot.

As mentioned above, the upper bound (2.10) consists of a term describing the parallel connection of the local contact resistances and another term due to the constriction of the heat flow density. It turns out that the constriction effect is mainly characterized by the correlation length l_c. For the limiting cases $a = 0$ and $a = 1$ the bound (2.10) becomes equal to the exact values w_e and w_i, respectively. The limit $w_e/w_i \to \infty$ is of particular interest. It means that there is no heat transfer outside the contact spots. Then equation (2.10) reduces to

$$w_+ = \frac{w_i}{a} + \frac{1-a}{a} l_c. \tag{2.16}$$

This bound is simply the sum of the film resistance of the contact spots and the constriction resistance which now only depends on the geometry of the contacts.

The upper bounds obtained in this section are based on very simple trial functions $\tilde{q}(\mathbf{R})$. It is possible to derive a more restricting upper bound without including more information about w_{st} than in bound (2.10), i.e. the two-point correlation function $\Gamma_w(\mathbf{R}')$. To this end we are going to construct a new, more appropriate trial function in the next section.

3. CONSTRUCTION OF A TRIAL FUNCTION

To derive a suitable trial function we start from equation (1.12) which determines the exact field $q(\mathbf{R})$. A formal solution may be obtained as follows [2, 3]. We first split up the operator $w_{st} + JP$ into a homogeneous non-stochastic operator L_0 and a stochastic term w'

$$L_0 := w_0 + JP, \qquad w'(\mathbf{R}) = w_{st}(\mathbf{R}) - w_0. \tag{3.1}$$

The constant parameter w_0 will be fixed later. Inserting the decomposition (3.1) into equation (1.12), we obtain

$$(L_0 + Pw')q = w_0\langle q\rangle = L_0\langle q\rangle. \tag{3.2}$$

The last equality holds because $P\langle q\rangle = 0$. Using operator calculus and denoting the inversion of an operator by the exponent '-1' and the unity operator by '1', we can transform equation (3.2) into

$$(1 - L_0^{-1} Pw')q = \langle q \rangle \qquad q = (1 + G_0 Pw')^{-1} \langle q \rangle, \quad G_0 := L_0^{-1}. \tag{3.3}$$

Equation (3.3) represents only a formal solution of equation (1.12) since the inversion of the operator $(1 + G_0 Pw')$ is unknown. In order to construct a practicable trial function we could replace G_0 by a number but this would lead back to an expression of the type (2.3). Therefore we first transform equation (3.3) into

$$q = (1 + G_0 Pw' - G_0 Pw')(1 + G_0 Pw')^{-1} \langle q \rangle = [1 - G_0 Pw' (1 + G_0 Pw')^{-1}] \langle q \rangle \tag{3.4}$$

and then replace the operator G_0 in the denominator by a constant parameter γ. The result is the trial function

$$\tilde{q} = (1 + G_0 PB) \langle q \rangle \tag{3.5}$$

with

$$B := -w'(1 + \gamma Pw')^{-1}. \tag{3.6}$$

Since in the following the operator B is only applied to a constant field $\langle q \rangle$ we may also write (see Appendix)

$$B \langle q \rangle = b \langle q \rangle \tag{3.7}$$

$$b := -w'(1 + \gamma w')^{-1} \langle (1 + \gamma w')^{-1} \rangle^{-1} \tag{3.8}$$

where b is not an operator but an ordinary function of position.

4. A GENERAL UPPER BOUND

Inserting the trial function (3.5) into the variational principle (1.11), we obtain the bound

$$w_+^{(1)} = \langle \tilde{q} \rangle^{-2} \langle [(1 + G_0 Pb) \langle \tilde{q} \rangle] \times (w' + L_0) [(1 + G_0 Pb) \langle \tilde{q} \rangle] \rangle. \tag{4.1}$$

Since the operator L_0, G_0 and P are self-adjoint in the sense of

$$\langle f_1(\mathbf{R}) L_0 f_2(\mathbf{R}) \rangle = \langle f_2(\mathbf{R}) L_0 f_1(\mathbf{R}) \rangle \tag{4.2}$$

where f_1 and f_2 are arbitrary functions of position, equation (4.1) may be transformed into

$$w_+^{(1)} = \langle L_0 + w' + 2Pb + 2w' G_0 Pb + bPG_0 Pb + bPG_0 w' G_0 Pb \rangle. \tag{4.3}$$

Because of the definitions (1.4) and (3.8), the relations

$$\langle P \ldots \rangle = 0 \tag{4.4}$$

and

$$\gamma w' Pb \rangle = \gamma w'(b - \langle b \rangle)) = (-w' - b)) \tag{4.5}$$

hold. With their help, expression (4.3) becomes, after some transformations,

$$w_+^{(1)} = \langle w_0 - b - b(G_0 - \gamma) Pb + bP(G_0 - \gamma) w' (G_0 - \gamma) Pb \rangle. \tag{4.6}$$

The last term on the RHS of equation (4.6) contains two integral operators G_0. Hence its calculation requires the knowledge of the three-point moment $\langle b(\mathbf{R}_1) w'(\mathbf{R}_2) b(\mathbf{R}_3) \rangle$ of the stochastic fields $b(\mathbf{R})$ and $w'(\mathbf{R})$. In most practical cases this information will hardly be available. But we can eliminate this term by choosing

$$w_0 = \max(w_{st}(\mathbf{R})), \quad w' = w_{st} - w_0 \leq 0. \tag{4.7}$$

Then the term becomes negative definite and if we omit it we obtain a new, greater upper bound

$$w_+ = w_0 - \langle b \rangle - \langle b(G_0 - \gamma) Pb \rangle \geq w_+^{(1)} \geq w_{\text{eff}}. \tag{4.8}$$

Let us recall that G_0 is an integral operator and, therefore, the last term reads in more detail

$$\langle b(G_0 - \gamma) Pb \rangle = \int d^2\mathbf{R}' \langle b(\mathbf{R}) [G_0(\mathbf{R} - \mathbf{R}') - \gamma \delta(\mathbf{R} - \mathbf{R}')] Pb(\mathbf{R}') \rangle. \tag{4.9}$$

The bound (4.8) involves only a two-point moment of the stochastic quantity b. The explicit form of the operator G_0 can easily be obtained in the Fourier representation. Let us define a function v by

$$v(\mathbf{R}) = G_0 Pb. \tag{4.10}$$

Its mean value vanishes

$$\langle v(\mathbf{R}) \rangle = G_0 \langle Pb \rangle = 0 \tag{4.11}$$

due to the definition of P(1.4). Applying the operator L_0 to equation (4.10) yields the integral equation

$$L_0 v(\mathbf{R}) = (w_0 + JP) v = (w_0 + J) v = Pb(\mathbf{R}) \tag{4.12}$$

for v. A Fourier transformation leads to

$$(w_0 + k^{-1}) v(\mathbf{k}) = Pb(\mathbf{k})$$
$$v(\mathbf{k}) = G_0(\mathbf{k}) Pb(\mathbf{k}) = (w_0 + k^{-1})^{-1} Pb(\mathbf{k}). \tag{4.13}$$

The kernel function $G_0(\mathbf{R})$ could be obtained from an inverse tansformation according to

$$G_0(\mathbf{R}) P = \int d^2\mathbf{k}\, e^{i\mathbf{kR}} G(\mathbf{k}) P = \int d^2\mathbf{k}\, e^{i\mathbf{kR}} (w_0 + k^{-1})^{-1} P. \tag{4.14}$$

But in the following only the Fourier representation $G_0(\mathbf{k})$, equation (4.13), will be used.

Let us recall that equation (4.8) represents a bound for arbitrary values of γ. In order to get an upper bound as low as possible we have to minimize expression (4.8) with respect to the open parameter γ. Using the self-adjointness of the corresponding operators and the identity

$$\frac{\partial}{\partial \gamma} \langle b \rangle = \langle bPb \rangle, \tag{4.15}$$

we obtain the following implicit equation which determines the optimal value of γ

$$\frac{\partial w_+}{\partial \gamma} = 2 \left\langle \frac{\partial b}{\partial \gamma} (G_0 - \gamma) Pb \right\rangle = 0. \tag{4.16}$$

We have to bear in mind that the function $b(\mathbf{R})$ also depends on γ. Therefore, only in special cases the optimal value of γ can be calculated explicitly from equation (4.16). In the next section we shall consider a binary model of w_{st}. In this case, the solution of equation (4.16) can easily be found.

5. BOUND FOR A BINARY MODEL

In the binary model we assume that according to equation (2.6), w_{st} takes only two distinct values w_i and w_e inside and outside the contacts, respectively. Then equation (4.7) requires

$$w_0 = w_e > w_i \tag{5.1}$$

and the function $b(\mathbf{R})$ given by equation (3.8) takes the simple form

$$b(\mathbf{R}) = b_i\Theta(\mathbf{R}) \tag{5.2}$$

$$b_i = (w_e - w_i)[1 - \gamma(1 - a)(w_e - w_i)]^{-1}. \tag{5.3}$$

Let us now calculate the parameter γ_0 determined by equation (4.16). Inserting equation (5.2) into equation (4.16), we find

$$2b_i\frac{\partial b_i}{\partial\gamma}[\langle\Theta(G_0 - \gamma)P\Theta\rangle] = 0. \tag{5.4}$$

In more detail this equation reads ($\gamma \to \gamma_0$)

$$\int d^2\mathbf{R}'G_0(\mathbf{R}')\langle\Theta(\mathbf{R})P\Theta(\mathbf{R}+\mathbf{R}')\rangle = \gamma_0\langle\Theta(\mathbf{R})P\Theta(\mathbf{R})\rangle$$

$$\gamma_0 = \frac{1}{a(1-a)}\int d^2\mathbf{R}'G_0(\mathbf{R}')[\langle\Theta(\mathbf{R})\Theta(\mathbf{R}+\mathbf{R}')\rangle - a^2]$$

$$= \int d^2\mathbf{R}'G_0(\mathbf{R}')\Gamma(\mathbf{R}') \tag{5.5}$$

where the correlation function $\Gamma(\mathbf{R}')$ is defined in equation (2.14). Using the Fourier transformation of the operator G_0 (4.13) and taking into account equation (5.1), we can rewrite equation (5.5) in the form

$$\gamma_0 = \frac{1}{4\pi^2}\int d^2\mathbf{k}(w_e+k^{-1})^{-1}\,\Gamma(\mathbf{k}),$$

$$\Gamma(\mathbf{k}) = \frac{1}{4\pi^2}\int d^2\mathbf{R}e^{i\mathbf{k}\mathbf{R}}\Gamma(\mathbf{R}). \tag{5.6}$$

To discuss the limit $w_e \to \infty$ it is more convenient to express γ_0 as follows

$$\gamma_0 = \frac{1}{w_e}\left\{\frac{1}{4\pi^2}\int d^2\mathbf{k}\left(1 - \frac{1}{1 + w_e k}\right)\Gamma(\mathbf{k})\right\}$$

$$= \frac{1}{w_e}\left\{\Gamma(\mathbf{R}=0) - \frac{\tilde{l}_c}{w_e}\right\} = \frac{1}{w_e}\left(1 - \frac{\tilde{l}_c}{w_e}\right) \tag{5.7}$$

with

$$\tilde{l}_c = \frac{1}{4\pi^2}\int d^2\mathbf{k}\frac{w_e}{1 + w_e k}\Gamma(\mathbf{k}). \tag{5.8}$$

The characteristic length $\tilde{l}_c$ depends on w_e but in the limit $w_e \to \infty$ it coincides with the geometrical quantity l_c already defined in equation (2.13) and which reads in Fourier representation

$$\left[\tilde{l}_c \to l_c = \frac{1}{4\pi^2}\int d^2\mathbf{k}\frac{1}{k}\Gamma(\mathbf{k}) \quad \text{for } w_e \to \infty.\right] \tag{5.9}$$

Now let us return to the calculation of the upper bound (4.8). From equations (5.4) and (5.2) it is easy to see that the term $\langle b(G_0-\gamma_0)Pb\rangle$ vanishes together with (5.4). Therefore, the bound reduces to

$$w_+ = w_e - \langle b(\mathbf{R})\rangle. \tag{5.10}$$

Inserting (5.2) into equation (5.10) we obtain, after some straightforward algebra,

$$w_+ = \frac{w_e[w_i+(1-a)(1-w_i/w_e)\tilde{l}_c]}{(1-a)w_i+aw_e+(1-a)(1-w_i/w_e)\tilde{l}_c}. \tag{5.11}$$

This upper bound is the main result of this section. It depends only on the two discrete values of the local contact resistance, on the area fraction of the contacts and on the parameter $\tilde{l}_c$ which is a functional of the correlation function Γ according to equation (5.8). For the limiting cases $a = 0$ and $a = 1$ equation (5.11) yields the exact values $w_+ = w_e$ and $w_+ = w_i$, respectively. In the limit $w_e = \infty$ the bound (5.11) coincides with the bound (2.16). For finite w_e the bound (5.11) proves to be always lower than the bound (2.16). In most cases the relations $w_e \gg l_c, w_i$ will be satisfied. Then an expansion of equations (5.8) and (5.11) yields

$$\tilde{l}_c = l_c - \frac{1}{2\pi w_e}\Gamma(\mathbf{k}=0)\ln\frac{w_e}{l_c} + O\left(\frac{l_c^2}{w_e}\right) \tag{5.12}$$

$$w_+ = \left(\frac{w_i}{a} + \frac{1-a}{a}l_c\right)\left(1 - \frac{1-a}{a}\frac{w_i + l_c}{w_e}\right)$$

$$- \frac{1-a}{a}\frac{l_c}{w_e}\left(w_i + \frac{\Gamma(\mathbf{k}=0)}{2\pi l_c}\ln\frac{w_e}{l_c}\right) + \ldots. \tag{5.13}$$

6. CIRCULAR CONTACT SPOTS

To check the usefulness of the bound obtained in equation (5.11) let us now calculate it for the special case of an arrangement of circular contact spots of equal radii. For this model an upper bound had already been derived in a previous paper [1] by the use of another trial function adapted to this special geometry. Moreover, this calculation had been restricted to the limiting values $w_i = 0$ and $w_e = \infty$. As discussed above, in this case the bound (5.11) reduces to the last term of equation (2.16) which we want to compare now to the result previously obtained in [1]. The contact area is described by a sum

$$\Theta(\mathbf{R}) = \sum_i \Theta^c(\mathbf{R} - \mathbf{R}_i) \tag{6.1}$$

$$\Theta^c(\mathbf{R}) = \begin{cases}0\\1\end{cases} \text{for } |\mathbf{R}| \gtrless R_0 \tag{6.2}$$

where Θ^c characterizes a single contact spot of radius R_0 and $\mathbf{R}_i$ denotes the centre of the ith contact. Inserting equation (6.1) in (2.13) we find the special correlation function

$$\Gamma(\mathbf{R}') = \frac{1}{a(1-a)}\left[\sum_{i,j}\langle\Theta^c(\mathbf{R}-\mathbf{R}_i)\Theta^c(\mathbf{R}+\mathbf{R}'-\mathbf{R}_j)\rangle - a^2\right]$$

$$= \frac{1}{a(1-a)}\left[\sum_i\langle\Theta^c(\mathbf{R}-\mathbf{R}_i)\Theta^c(\mathbf{R}+\mathbf{R}'-\mathbf{R}_i)\rangle\right.$$

$$+ \sum_{i \neq j} \langle \Theta^c(\mathbf{R}-\mathbf{R}_i)\Theta^c(\mathbf{R}+\mathbf{R}'-\mathbf{R}_j)\rangle - a^2 \Bigg]. \quad (6.3)$$

Due to the definition of the angular brackets (1.5) the average $\langle \Theta^c(\mathbf{R}-\mathbf{R}_i)\Theta^c(\mathbf{R}+\mathbf{R}'-\mathbf{R}_j)\rangle$ depends only on the difference vector $(\mathbf{R}'+\mathbf{R}_i-\mathbf{R}_j)$. Considering a very large area $A \to \infty$ with a great number of single contacts $N \to \infty$ $(N/A = n = \text{const.})$, we may replace the sum over the pairs $i \neq j$ by an integration over the difference vector $(\mathbf{R}_i - \mathbf{R}_j) \to \mathbf{R}_d$ weighted by the pair distribution function $p(\mathbf{R}_d)$ normalized to $(N - 1)$. In the following instead of p we shall use the pair correlation function $g(\mathbf{R}_d)$ of the contacts which is related to $p(\mathbf{R}_d)$ by

$$p(\mathbf{R}_d) = n[1 + g(\mathbf{R}_d)] \quad (6.4)$$

so that g vanishes at infinity. Now equation (6.3) can be written as

$$\begin{aligned}\Gamma(\mathbf{R}') = \frac{1}{a(1-a)}\,[&N\langle \Theta^c(\mathbf{R})\Theta^c(\mathbf{R}+\mathbf{R}')\rangle \\ &+ N \textstyle\int \mathrm{d}^2\mathbf{R}_d n(1+g(\mathbf{R}_d)) \\ &\times \langle \Theta^c(\mathbf{R})\Theta^c(\mathbf{R}+\mathbf{R}+\mathbf{R}_c)\rangle - a^2] \end{aligned} \quad (6.5)$$

or, according to equation (1.5)

$$\begin{aligned}\Gamma(\mathbf{R}') = \frac{1}{a(1-a)}\,[&n \textstyle\int \mathrm{d}^2\mathbf{R}\Theta^c(\mathbf{R})\Theta^c(\mathbf{R}+\mathbf{R}') \\ &+ n^2 \textstyle\int \mathrm{d}^2\mathbf{R}_d \int \mathrm{d}^2\mathbf{R}(1 + g(\mathbf{R}_d)) \\ &\times \Theta^c(\mathbf{R})\Theta^c(\mathbf{R}+\mathbf{R}'+\mathbf{R}_d) - a^2]. \end{aligned} \quad (6.6)$$

This equation contains convolution integrals which may be reduced to products by a Fourier transformation. Then, using

$$a = n\pi R_0^2 \quad (6.7)$$

and

$$\int \mathrm{d}^2\mathbf{R} e^{i\mathbf{k}\mathbf{R}}\, \Theta^c(\mathbf{R}) = \frac{2\pi R_0}{k} J_1(kR_0) \quad (6.8)$$

where J_1 is the Bessel function of first order, we obtain

$$\Gamma(\mathbf{k}) = \frac{1}{1-a}\frac{4\pi}{k^2} J_1^2(kR_0)\left[1 + \frac{a}{\pi R_0^2} g(\mathbf{k})\right]. \quad (6.9)$$

If equation (6.9) is inserted into expression (5.9) for l_c, the bound (2.16) becomes for $w_i = 0$

$$\begin{aligned} w_+ &= \frac{2R_0}{a}\int_0^\infty \mathrm{d}x \frac{J_1^2(x)}{x^2} + \frac{1}{\pi^2 R_0^2}\int \mathrm{d}^2\mathbf{k}\frac{J_1^2(kR_0)}{k^3} g(\mathbf{k}) \\ &= \frac{8}{3\pi}\frac{R_0}{a} + \frac{1}{R_0}\int \mathrm{d}^2\mathbf{R}\varphi\left(\frac{R}{R_0}\right) g(\mathbf{R}) \end{aligned} \quad (6.10)$$

where

$$\varphi(\rho) := \frac{2}{\pi}\int_0^\infty \mathrm{d}x \frac{J_1^2(x)}{x^2} J_0(x\rho). \quad (6.11)$$

The bound obtained in [1] has a completely analogous form. To distinguish it from the present result we designate it by an index 1

$$(w_+)_1 = \frac{\pi}{4}\frac{R_0}{a} + R_0 \int \frac{\mathrm{d}^2\mathbf{R}}{R_0^2}\varphi_1\left(\frac{R}{R_0}\right) g(\mathbf{R}) \quad (6.12)$$

where

$$\begin{aligned}\varphi_1(\rho) := \frac{1}{4\pi^2}\int &\mathrm{d}^2\rho'(1-\rho'^2)^{-1/2}\,\Theta^c(\rho') \\ &\times \left[\frac{\pi}{2}\Theta^c(\rho-\rho') + (1 - \Theta^c(\rho-\rho')\right. \\ &\left.\times \arcsin(|\rho-\rho'|^{-1})\right]. \end{aligned} \quad (6.13)$$

For small area fractions of the contacts $a \ll 1$ the first terms in the bounds (6.10) and (6.12) become predominant. They correspond to the parallel connection of isolated circular contact spots. In the limit $a \to 0$ the first term of (6.12) gives the exact result because the exact heat flow distribution of an isolated circular contact

$$q(\mathbf{R}) = \begin{cases} q_0(1 - R^2/R_0^2) \\ 0 \end{cases} \text{for } R \lessgtr R_0 \quad (6.14)$$

had been used to derive this bound. Contrarily, the bound (6.10) has been calculated by assuming a constant heat flow density inside the contact spot according to equation (2.3). Despite this crude approximation the first term in (6.10) lies only 8% above the corresponding exact contribution in equation (6.12).

The second terms in the bounds containing the correlation function g differ only little from one another. This can be seen from Fig. 1 representing the functions φ and φ_1. Their maximal difference amounts about 10% for small values of ρ and their asymptotic behaviour for $\rho \to \infty$ is the same. In order to illustrate the influence of these terms for finite area fractions, the bounds (6.10) and (6.12) have been calculated for a triangular lattice of circular contact spots. The results, multiplied by the factor $4a/\pi R_0$, are plotted in Fig. 2.

In general the bound w_+ lies above the bound $(w_+)_1$ previously obtained. This is not surprising because, in order to derive $(w_+)_1$, the special shape of the contact spots has been taken into account from the beginning, whereas the bound w_+ contains information about the geometry only in form of the two-point correlation function Γ which does not uniquely characterize the shape of the contact spots.

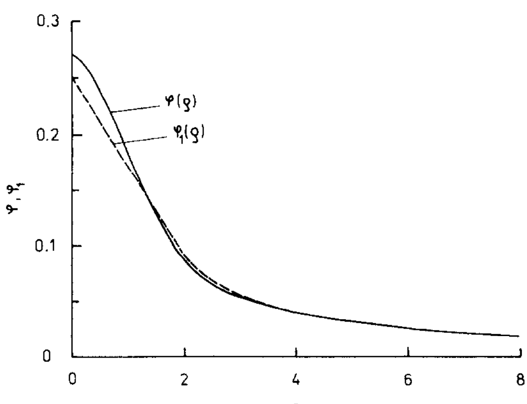

Fig. 1. Numerical calculation of the functions φ and φ_1 defined by equations (6.11) and (6.13), respectively.

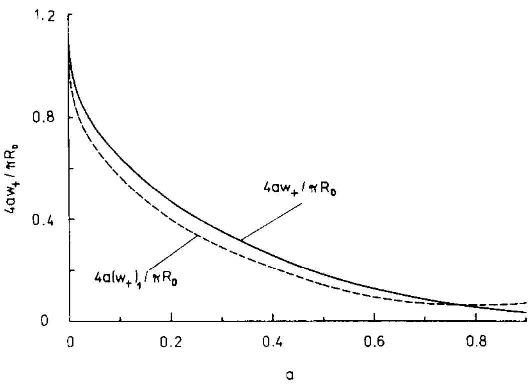

FIG. 2. Upper bounds w_+ (6.10) and $(w_+)_1$ (6.12) (multiplied by the factor $4a/\pi R_0$) for a triangular lattice of circular contact spots as function of the area fraction a.

7. CONCLUSION

Starting from a variational principle given in a previous work [1], we have calculated upper bounds for the macroscopic or effective contact resistance. By the use of a simple trial function for the heat flow density we could obtain a bound which involves only the two-point correlation function of the random local contact resistance. It consists of two terms. The first one corresponds to the parallel connection of the local contact resistances. The second one is a functional of the two-point correlation function and may be interpreted as an estimation of the constriction effect. A more restrictive bound without including more information about the local contact resistance could be derived by means of a more sophisticated trial function. This bound has been discussed for the special case of a binary model in which the local contact resistance takes only two distinct values inside and outside the contact spots, respectively. If the local contact resistance outside the contacts is very high, the two bounds differ only slightly from one another. To test the quality of the general bound derived in this paper, we have applied it to the special geometry of an arbitrary arrangement of identical circular contact spots and then compared the result to a special bound derived previously for this case. The agreement is satisfactory.

In order to apply the bound derived above we need the values of the local contact resistance as well as the two-point correlation function of the contact area distribution. Both must be taken from experiments or from other considerations. To get a satisfactory estimate of the effective contact resistance it would be desirable to find also lower bounds. But, unfortunately, the derivation of lower bounds seems not to be possible without very special assumptions about the geometry of the contact spots.

APPENDIX

Proof of equation (3.8)

Let us set

$$v = (1 + \gamma P w')^{-1} \langle q \rangle \tag{A1}$$

or

$$(1 + \gamma P w')\, v = (1 + \gamma w')\, v - \gamma \langle w' v \rangle = \langle q \rangle . \tag{A2}$$

Because of $\langle P \ldots \rangle = 0$ an average over (A2) yields

$$\langle v \rangle = \langle q \rangle . \tag{A3}$$

From (A2) we find

$$v = (1 + \gamma w')^{-1} (\langle q \rangle + \gamma \langle w' v \rangle) \tag{A4}$$

and

$$\langle v \rangle = \langle (1 + \gamma w')^{-1} (\langle q \rangle + \gamma \langle w' v \rangle) \rangle . \tag{A5}$$

Dividing (A4) by (A5) and considering (A3) gives

$$v = (1 + \gamma P w')^{-1} \langle q \rangle$$
$$= (1 + \gamma w')^{-1} \langle (1 + \gamma w')^{-1} \rangle^{-1} \langle q \rangle . \tag{A6}$$

Apart from a factor $(-w')$ this agrees with relation (3.8).

REFERENCES

1. M. Bobeth and G. Diener, Variational bounds for the effective thermal contact resistance between bodies with rough surfaces, *Int. J. Heat Mass Transfer* **25**, 111–117 (1982).
2. P. H. Dederichs and R. Zeller, Variational treatments of the elastic constants of disordered materials, *Z. Physik* **259**, 103–116 (1973).
3. E. Kröner and H. Koch, Effective properties of disordered materials, *SM Archives* **1**, 183–238 (1976).

LIMITE SUPERIEURE DE LA RESISTANCE DE CONTACT THERMIQUE EFFECTIVE ENTRE DEUX CORPS A SURFACES RUGUEUSES

Résumé—Dans un article précédant [*Int. J. Heat Mass Transfer* **25**, 111–117 (1982)] des principes variationels ont été donnés, à partir desquels des limites inférieures et supérieures de la résistance de contact effective peuvent être déduites. Ici on utilise cette méthode pour déterminer des limites supérieures pour une distribution et des formes arbitraires des taches de contact. Les limites dépendent de la fonction de correlation binaire de la résistance de contact locale. En particulier, on considère un modèle binaire n'admettant que deux valeurs discrètes de la résistance de contact locale, à savoir à l'intérieur et à l'extérieur des taches de contact. La résistance de constriction est surtout caractérisée par une longueur de correlation des taches de contact. Les résultats obtenus sont comparés à la limite spéciale pour des taches circulaires donnée dans l'article précédant.

OBERE GRENZEN FÜR DEN EFFEKTIVEN WÄRMEÜBERGANGSWIDERSTAND ZWISCHEN KÖRPERN MIT RAUHEN OBERFLÄCHEN

Zusammenfassung—In einer früheren Arbeit [*Int. J. Heat Mass Transfer* **25**, 111–117 (1982)] wurden Variationsprinzipien angegeben, mittels derer obere und untere Grenzen für den effektiven Wärmeübergangswiderstand berechnet werden können. In vorliegender Arbeit wird diese Methode zur Bestimmung oberer Grenzen bei beliebiger Form und Verteilung der einzelnen Kontakte angewandt. Die Grenzen ergeben sich als Funktionale der Zweierkorrelationsfunktion des lokalen Kontaktwiderstandes. Insbesondere wurde ein binäres Modell betrachtet, in welchem der lokale Kontaktwiderstand nur zwei diskrete Werte innerhalb bzw. außerhalb der Kontaktflächen besitzt. Der Konstriktionswiderstand wird im wesentlichen durch eine Korrelationslänge der Kontaktflächen charakterisiert. Die Ergebnisse werden mit der in der früheren Arbeit speziell für kreisförmige Kontakte erhaltenen Grenze verglichen.

ВЕРХНИЕ ГРАНИЦЫ ЭФФЕКТИВНОГО КОНТАКТНОГО ТЕПЛОСОПРОТИВЛЕНИЯ МЕЖДУ ТЕЛАМИ С ШЕРОХОВАТЫМИ ПОВЕРХНОСТЯМИ

Аннотация — В предыдущей работе [I, *Int. J. Heat Mass Transfer*, **25**, 111–117 (1982)], вариационные принципы были использованы для получения нижних и верхних границ эффективного контактного теплосопротивления. В настоящей работе используется этот метод для вычисления верхних границ при любой геометрии и распределения областей соприкосновения. Границы являются функционалами парной корреляционной функции локального контактного сопротивления. В настности была рассмотрена бинарная модель, в которой существуют только два значения локального контактного сопротивления либо вне либо внутри контактных областей. Сопротивление стягивания характеризуется корреляционной длиной областей соприкосновения. Результаты сравниваются со специальной границей для круглых областей соприкосновения, полученой в [I].